

Comment on the “Influence of Cosmological Transitions on the Evolution of Density Perturbations”

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Abstract

A recent paper by Martin and Schwarz [1] argues that the “standard inflationary result” has been finally proven. The result itself is formulated as: “the closer the inflationary epoch is to the de Sitter space-time, the less important are large-scale gravitational waves in the CMBR today”. Beginning from the basic equations of Grishchuk [2], [3] the authors say [1] that Grishchuk’s conclusion about approximate equality of metric amplitudes for gravitational waves and density perturbations “is wrong because the time evolution of the scalar metric perturbation through the (smooth) reheating transition was not calculated correctly”. They reiterate a claim about “big amplification” of scalar perturbations (in contrast to gravitational waves) during reheating. The authors say [1] that after appropriate correction they have recovered the “standard result” within Grishchuk’s approach. It is shown in this Comment that the “big amplification” is a misinterpretation. There is no difference in the evolution of long-wavelength metric perturbations for gravitational waves and density perturbations: they both stay approximately constant. The influence of cosmological transitions on the evolution is none at all, as long as the wavelength of the perturbation is much larger than the Hubble radius. It is shown that from the approach of [2, 3] follow the conclusions of [2, 3] without change. Finally, it is argued that the “standard inflationary result” does not follow from the correct evolution and quantum normalization of density perturbations.

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Cosmological gravitational waves and density perturbations could have been generated as a result of the superadiabatic (parametric) amplification of their zero-point quantum oscillations by strong variable gravitational field of the early Universe. The theoretical predictions regarding the amplitudes and spectral slopes of these primordial perturbations are of a considerable practical importance, especially in view of the forthcoming sensitive observations.

One possibility is that the early Universe was driven by a scalar field φ , which has latter decayed into a normal matter. For this model, some theoretical studies of density perturbations have led to the what is known as the “standard inflationary result”. The central inflationary formula for the generated matter density variation is usually written as

$$(\delta\rho/\rho)_f \sim (H^2/\dot{\varphi})_i$$

where the left-hand-side is what we want to know at the end of the long-wavelength regime (second Hubble radius crossing), and the right-hand-side is what we need to evaluate at the beginning of the long-wavelength regime (first Hubble radius crossing). The same formula is often written as

$$(\delta\rho/\rho)_f \sim (V^{3/2}/V')_i$$

where $V(\varphi)$ is the scalar field potential and $V' = \frac{\partial V}{\partial \varphi}$.

To say the least, this formula looks very counter-intuitive. Imagine that the potential $V(\varphi)$ has an inflection point, where $V' = 0$, and the slowly-rolling scalar field passes at a certain moment of time through that point. Then, the final amplitude $(\delta\rho/\rho)_f$ of the mode, which entered the long-wavelength regime at that moment of time, is predicted to be infinite. Since the amplitude of the metric perturbation h_f at the second crossing is of the same order of magnitude as the matter density variation $(\delta\rho/\rho)_f$ (both are dimensionless numbers), the metric (curvature) perturbation is also predicted to be infinite. Moreover, for the generation of a spectral interval with the almost Harrison-Zeldovich slope $n = 1$ one needs an interval of the almost de Sitter expansion, that is, the potential $V(\varphi)$ should be very flat, $V' \approx 0$, at some interval of φ . Then, all the amplitudes of the generated density perturbations in this interval of spectrum are predicted, according to the “standard” formula, to be arbitrarily large. In addition, all these predictions are in a sharp conflict with the generation of gravitational waves, whose final amplitudes are finite and small, even if the discussed interval of the early expansion was of the de Sitter type.

There is not any obvious physical reason for the appearance of this divergent result in a space-time (gravitational pump field) with finite and relatively small background curvature (the Hubble parameter H_i is probably 5 orders of magnitude smaller than the Planckian value) which sets the numerical level of the nominator in the “standard” formula. This result is also in conflict with the finite “temperature” [4] of the de Sitter space-time, which is determined by its constant Hubble parameter H . Whatever “particles” are being produced, they are supposed to have finite and small energy in every frequency interval. However, according to the “standard” formula, even a short interval of the de Sitter evolution is capable of generating an arbitrarily large amount of “particles” - density (curvature) perturbations.

The results of calculations of ref. [2, 3] have shown that the “standard” formula does not follow from the proper evolution and quantum normalization of density perturbations.

Although the calculations are quite involved technically and include elements of quantum mechanics, the bottom line is simple and can be summarised as follows. Cosmological density perturbations are not perturbations in the matter component only, be it a scalar field or a perfect fluid; they necessarily include the metric (gravitational field) perturbations, and do not exist without them. We are mostly interested in the evolution of the metric perturbations, because they are the carriers of density perturbations - the scalar field, together with its perturbations, is supposed to be converted into other forms of matter anyway. The evolution of the metric perturbations is trivial: the dominant solution is a constant, as long as the wavelength is much larger than the Hubble radius, and independently of what kind of transformations of the matter content are taking place during all the time of expansion from the beginning of the long-wavelength regime and up to its end. So, the final amplitude of the metric perturbation is practically the same as the initial one, $h_f \approx h_i$. To predict the final amplitude we need to know the initial amplitude. Classically, the initial amplitude is in our hands and is determined by our choice of initial conditions. Quantum-mechanically, the minimal initial amplitude h_i is determined by quantum fluctuations. To find the numerical value of h_i we certainly cannot rely (in contrast to what is often being done) on the quantization of a free scalar field, which is not accompanied by metric perturbations at all. We need to quantize the “fundamental” field, which fully represents cosmological density perturbations and includes metric perturbations. The dynamical equation for this “fundamental” field is very similar (as well as the Lagrangian, Hamiltonian and other structures) to the dynamical equation for cosmological gravitational waves, and they are exactly the same in case of any of the power-law expansions. The initial amplitude h_i , determined by the quantum fluctuations for a given mode with wavelength λ , is $h_i \approx l_{Pl}/\lambda_i$, where $\lambda_i = c/H_i$ and H_i is the Hubble parameter at the first crossing. This finite and small number effectively translates into h_f and $(\delta\rho/\rho)_f$, that is, $(\delta\rho/\rho)_f \approx h_f \approx h_i$. Since the initial amplitude for gravitational waves (derived from its own quantum considerations) is also approximately equal to h_i , and since it is being transmitted without change to the end of the long-wavelength regime, the final amplitudes of gravitational waves and density perturbations are also approximately equal. It follows from the calculations of ref. [2, 3] that in the correct version of the “standard” formula the right-hand-side of this formula must be multiplied by the dimensionless factor $(\sqrt{-\dot{H}/H^2})_i$. This factor cancels out the zero in the denominator and makes the amplitudes of density perturbations finite and of the same order of magnitude as the amplitudes of gravitational waves.

Martin and Schwarz [1] (as well as some of their predecessors) disagree with the outlined above and used in [2, 3] law of evolution for the metric perturbations associated with density perturbations. They claim that the long-wavelength scalar metric perturbations have experienced, in contrast to the metric perturbations representing gravitational waves, “big amplification” during “reheating”, that is, during a short interval of time when the scalar field decayed into the radiation-dominated matter. In other words, they try to explain the huge number for h_f suggested by the “standard” formula by the evolution of a small initial number which was determined by quantum fluctuations. This seems to be the only way to make sense of the “standard” formula. Otherwise, if you agree with the law $h_f \approx h_i$, the huge h_i must be postulated from the very beginning, and this is certainly not the amplitude dictated by the zero-point quantum oscillations of a quantized field. The evolution of the long-wavelength metric perturbations is the main point of the disagreement, and we will

address it below.

Let us formulate the problem more precisely. We consider classical solutions to the perturbed Einstein equations. The equation of state of the background matter may vary as a function of time. A known example is the decoupling transition from the radiation-dominated era to the matter-dominated era. The decoupling transition is not an issue of much disagreement, so we will concentrate on the reheating transition from the era governed by the scalar field to the era governed by the radiation. Theoretically, the transition could be either smooth, so that the scale factor, its first time derivative, and its second time derivative are continuous; or sharp, so that the second time derivative of the scale factor (and the first time derivative of the Hubble parameter) experience a finite jump. Other quantities of the background solution are either nonimportant or calculable. We consider density perturbations (scalar eigen-functions in Lifshitz's classification) and gravitational waves (tensor eigen-functions in Lifshitz's classification).

The question is: What can be stated about the evolution of metric perturbations in the long-wavelength regime? The provisional answer to this question, even if not for all conceivable cases, is effectively known since the pioneering paper of Lifshitz [5]: the dominant solution is a constant. The perturbation in the matter energy density is slowly growing, all the time from the first Hubble radius crossing to the second Hubble radius crossing. But the perturbation in the metric tensor remains constant, both, for density perturbations and for gravitational waves. So, if we have started deeply in the very early Universe with the perturbed metric amplitude at the level, say, 10^{-4} , we will get the same number 10^{-4} at the end of the long-wavelength regime. The end of this regime is today for wavelengths of the order of today's Hubble radius. And the initial metric amplitude, say, 10^{-20} also translates into 10^{-20} today, both for gravitational waves and density perturbations, and nothing dramatic happens on the way from one crossing to another. The perturbation in the matter energy density (in case of density perturbations) will grow, but only up to the level 10^{-4} or, correspondingly, 10^{-20} .

This answer requires a few clarifications. Which metric perturbations are we working with? After a few attempts one usually realizes that the class of the battle-tested synchronous (freely falling) coordinate systems is a very convenient choice for practical calculations. There exists some remaining freedom in the choice of a synchronous coordinate system. This freedom is an advantage rather than disadvantage, because it allows us, if we want, to adjust the coordinate system in such a manner that by the time of reaching the matter-dominated stage it will become a comoving coordinate system in addition to already being synchronous - the most favorite choice. However, we will mostly work with a single function, denoted below as $\bar{\mu}/a$, which completely describes the metric perturbations, and which is independent of the remaining freedom. It is for this function we gave the estimate of h_i and $h_f \approx h_i$. In general, there are two essential metric components (two polarisation states) for each type of perturbations. These components are independent functions in case of gravitational waves. But for the density perturbations, that we actually consider, one polarisation component (longitudinal-longitudinal) is not independent and can be expressed in terms of the another (scalar) component. From a given complete solution one can calculate any of the so-called "gauge-invariant" variables, and we will do this calculation below for some of them.

As was already mentioned, the dominant ("growing") long-wavelength solution for each

independent metric component, both in case of gravitational waves and density perturbations, is a constant C . For simple numerical evaluations, it is convenient to operate with a single number - the characteristic amplitude - which is the square root of sum of squares of the two polarization components. In practice, the characteristic amplitude is again C . For the perturbations with wavelengths of the order of today's Hubble radius, this number C gives also a rough numerical estimate [6], [7] of the produced quadrupole anisotropy in the CMB radiation: $\delta T/T = C$. And, finally, if this constant C is approximately the same number for gravitational waves and for density perturbations, the anisotropies $\delta T/T$ induced by gravitational waves and by density perturbations will also be approximately equal. Why these numbers C should be approximately equal for primordial density perturbations and for primordial gravitational waves is determined by the initial conditions. This is the only one place where quantum considerations relate to our discussion. We have mentioned this point above and will return to it later, but it is not in the focus of our attention to the claimed dramatic "influence of transitions" in the case of density perturbations.

In a sense, we know in advance the correct answer to the question of evolution of long-wavelength metric perturbations: the dominant solution is a constant. This expected answer was essentially rederived in great detail in [2] and has led to the conclusion that the "standard" result does not follow from the correct evolution and quantum normalization of density perturbations. The work [2, 3] had been several times criticized and "refuted" by members of inflationary community. Martin and Schwarz [1] make an analysis of two previous dedicated papers [8], [9], and we will need their summary. The authors of [1] note that "according to Deruelle and Mukhanov, Grishchuk made two mistakes: he took wrong joining conditions and he used the wrong equation of state...at the reheating transition". However, Martin and Schwarz [1] conclude that "both his joining conditions at the reheating transition and the equation of state after reheating have been used correctly". Martin and Schwarz [1] recall that according to Caldwell, Grishchuk's conclusion occurs because the long-wavelength limit $n^2 \rightarrow 0$ "has not been taken consistently". However, they conclude, "the limit had been taken properly" by Grishchuk. So, we can now safely put aside other papers and concentrate on the "final proof" [1] of the statement that, nevertheless, Grishchuk's conclusion is wrong because of the overlooked "big amplification" of the metric perturbations during reheating.

We need some equations and relationships. Most of them are reproduced in [1] but it is better to have them, so to say, from the first hands.

We write the perturbed metric in the form

$$ds^2 = -a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j].$$

We will often use two new functions of the scale factor $a(\eta)$: $\alpha = a'/a$ and $\gamma = 1 - \alpha'/\alpha^2 = 1 + (a/a')'$. In terms of t time, $cdt = a(\eta)d\eta$, the function γ is $\gamma = -\dot{H}/H^2$ where H is the Hubble parameter $H = \dot{a}/a = c\alpha/a$. As a consequence of the background Einstein equations, we have $1 + p_0/\epsilon_0 = (2/3)\gamma$. For models under consideration, the instantaneous equation of state may vary, smoothly or sharply, from $p_0 = -\epsilon_0$ ($\gamma = 0$) to $p_0 = \epsilon_0/3$ ($\gamma = 2$) and later to $p_0 = 0$ ($\gamma = 3/2$).

In case of density perturbations, we write a spatial Fourier component of the metric perturbations in the form

$$h_{ij} = h(\eta)\delta_{ij}Q + h_l(\eta)\frac{1}{n^2}Q_{,i,j}$$

where $Q = e^{i\mathbf{n}\cdot\mathbf{x}}$ or $e^{-i\mathbf{n}\cdot\mathbf{x}}$. The function $h(\eta)$ represents the scalar polarisation state, whereas the $h_l(\eta)$ represents the longitudinal-longitudinal polarisation state.

It was shown [2, 3] that, for models governed by scalar fields and perfect fluids, the full set of perturbed Einstein equations can be reduced to a single second-order differential equation (master equation). To write this equation specifically for models governed by a scalar field with arbitrary scalar field potential, we introduce the function $\bar{\mu}/a$:

$$\frac{\bar{\mu}}{a} = h + \frac{1}{\alpha\gamma}h' \quad (1)$$

and the function μ : $\mu = \sqrt{\gamma}\bar{\mu}$. Then, the mentioned master equation takes the form

$$\mu'' + \mu \left[n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] = 0. \quad (2)$$

The function $\bar{\mu}/a$ can be called the “residual-gauge-invariant” part of $h(\eta)$. This function plays the central role in our discussion.

All the functions describing the perturbations - $h(\eta)$, $h_l(\eta)$, and $\varphi_1(\eta)$, where $\varphi_1(\eta)$ is the scalar field perturbation, $\varphi = \varphi_0(\eta) + \varphi_1(\eta)Q$ - can now be found from solutions to eq. (2). In particular, using (1) we find

$$h(\eta) = \frac{\alpha}{a} \left[\int_{\eta_0}^{\eta} a\gamma \left(\frac{\bar{\mu}}{a} \right) d\eta + C_i \right] \quad (3)$$

where C_i is an arbitrary integration constant reflecting the remaining coordinate freedom.

Equation (2) has been presented, on purpose, in the form similar to the one of the previously explored equation for gravitational waves [10]:

$$\mu'' + \mu \left[n^2 - \frac{a''}{a} \right] = 0 \quad (4)$$

where, in case of gravitational waves,

$$h_{ij} = \frac{\mu}{a} Q_{ij} \quad (5)$$

and $Q_{ij} = p_{ij}e^{i\mathbf{n}\cdot\mathbf{x}}$, $p_{ij}\delta^{ij} = 0$, $p_{ij}n^j = 0$ for each of two independent polarisation tensors p_{ij} . (Equations (2) and (4) are reproduced as (3.24) and (2.16) in [1].)

Equations and solutions for the function μ representing density perturbations can be found from equations and solutions for the function μ representing gravitational waves by the simple replacement $a \rightarrow a\sqrt{\gamma}$ [2].

We are now in the position to find long-wavelength solutions to (4) and (2). It is known [10] that as long as the wave stays “under the barrier”, that is $n^2 \ll |a''/a|$, the approximate solution to (4) is

$$\mu = C_1 a + C_2 a \int \frac{d\eta}{a^2} \quad (6)$$

where C_1 and C_2 are arbitrary constants. Combining (6) with (5) and neglecting the decaying term with C_2 , we find the dominant solution for the gravity-wave characteristic amplitude

$$h^{g.w.} = C_1. \quad (7)$$

In other words, independently of the sort of matter which drives the scale factor $a(\eta)$ of our model, the dominant solution is a constant during all the time that the wavelength is longer than the Hubble radius. (A certain distinction between the notions of the “barrier” and the Hubble radius “crossing” is not important for our discussion.)

Similarly to the gravity-wave case, the long-wavelength solution to eq. (2) is [2]:

$$\bar{\mu} = C_1 a + C_2 a \int \frac{d\eta}{a^2 \gamma}, \quad (8)$$

and the dominant part is $\mu \propto a\sqrt{\gamma}$, that is,

$$\frac{\bar{\mu}}{a} = C_1 \quad (9)$$

and $h_i = h_f = C_1$. Again, the scale factor $a(\eta)$ and the function $\gamma(\eta)$ change with time, and, in particular, the function $\gamma(\eta)$ may be changing from very small values up to $\gamma = 2$, but solution (9) is still valid. Using (9) and the fact that $a\gamma \equiv (a/\alpha)'$, we find from (3):

$$h(\eta) = C_1 + \frac{\alpha}{a} C_i.$$

As was explained above, the term with C_i is useful, but in any case this term is decaying and we neglect it for the purposes of our discussion. It can also be shown [2] that the component $h_l(\eta)$ slowly varies but is small all the time, and can reach the numerical level C_1 only at the end of the long-wavelength regime. So, in case of density perturbations, similarly to eq. (7) for gravitational waves, the dominant solution for the characteristic metric amplitude is a constant

$$h^{d.p.} = C_1. \quad (10)$$

This is true independently of whether the scale factor $a(\eta)$ is driven by a scalar field or by a perfect fluid. Of course, the constancy of the metric amplitude at the radiation-dominated and matter-dominated stages is known for long time [11].

The problem of the evolution admits also a complete solution in case of a sharp transition. Imagine that the function $\gamma(\eta)$ is discontinuous at the transition point and takes the values γ_l and γ_r to the left and to the right of the transition point. The γ_l can be arbitrarily close to zero, and γ_r can be 2. It was shown [2] that eq. (2) requires the continuity of the function v , where

$$v = \gamma \left(\frac{\bar{\mu}}{a} \right)' \quad (11)$$

and, hence, the continuity of the function $\bar{\mu}/a$. The continuity of $\bar{\mu}/a$ can be derived either directly from (2) or from inspection of (11). Indeed, a jump of $\bar{\mu}/a$ would develop a δ - function which could not be compensated by the step function γ in the right-hand-side of (11), and hence the continuity of v would be violated. Instead, eq. (11) requires a finite jump in the first derivative of $\bar{\mu}/a$. The value of this jump is determined by the jump of γ and by the continuity of v [12]. Obviously, the continuity of $\bar{\mu}/a$ guarantees that the constant

C_1 is being carried over any transition, including the reheating transition from $p_0 = -\epsilon_0$ to $p_0 = \epsilon_0/3$, without any changes. The “influence of reheating”, whether the transition is smooth or sharp, on the scalar metric perturbations is none at all.

At this point it is necessary to say that the formula $\mu|_{\eta=\eta_1-0} = \mu|_{\eta=\eta_1+0}$ printed at p. 7161 of [2] is indeed an error (which could have caused trouble for Martin and Schwarz). [There is also another obvious error on the same line: $\sqrt{(2+\beta)/(1+\beta)}$ should read $(2+\beta)/(1+\beta)$.] In the joining condition, it was meant to be $\bar{\mu}$, not μ , but the error arose because of the change of notations in original versions of the manuscript. It is obvious that this error had not been used and had not influenced the calculations. It is clearly stated at the very end of p. 7164 which condition had actually been used: “From the continuous joining of $h(\eta)$...one derives...”, which implies the continuous joining of the function $\bar{\mu}/a$, not μ/a . The full set of conditions is reproduced as eq. (5.11) in ref. [1], which, the authors agree [1], “are exactly the conditions that have been used” in ref. [2]. So, there was no mistake in calculations, caused by the mentioned error at p. 7161 of [2], contrary to what the authors of [1] suspect in their Sec. VI. Their eq. (6.5) is correct and the same as was actually used in [2], but it does not lead to any overlooked “big amplification” as we shall see below.

We have shown that the evolution of scalar metric perturbations is as simple as the evolution of tensor metric perturbations. This evolution is at the basis of practical calculations in [2, 3]. There is no room for a “big amplification”, which would pick up specially the metric amplitude of density perturbations and increase it, say, from 10^{-20} to 10^{-4} , while leaving the metric amplitude of gravitational waves at the initial level 10^{-20} . We gave the complete solution to the problem of evolution and did not need any concept of the gauge-invariant formalism or ζ conservation law. But in order to demonstrate that the claimed “big amplification” is a misinterpretation, we need to deal with those concepts.

Martin and Schwarz [1] work with the gauge-invariant potentials Φ and ζ . The important role is allocated to the “constancy of ζ ”. The quantity ζ is defined by their eq. (4.8):

$$\zeta = \frac{2}{3} \frac{\mathcal{H}^{-1}\Phi' + \Phi}{1+w} + \Phi, \quad (12)$$

where $\mathcal{H} = \alpha$ and $1+w = 1+p_0/\epsilon_0 = (2/3)\gamma$.

Since we have solved our perturbation problem, any quantity, including any gauge-invariant potential, can be calculated. It was shown [2b, 3] that the definition of Φ (see (2.3) in [1]), plus available Einstein equations, produces the exact equality

$$\Phi = \frac{1}{2n^2} \alpha \gamma \left(\frac{\bar{\mu}}{a} \right)' \quad (13)$$

(reproduced as (3.27) in [1]). It was also shown [2b, 3] that the definition (12), plus (13) and, most importantly, plus the master equation (2), produces the exact equality

$$\zeta = -\frac{1}{2} \left(\frac{\bar{\mu}}{a} \right). \quad (14)$$

In other words, after properly taking into account all the equations, the quantity ζ turns out to be simply the function $\bar{\mu}/a$ (up to the factor $-(1/2)$) which we are working with. Equality (14) demonstrates that the ζ conservation law is empty, in the sense that it does not give us anything new above what we have already derived in a much simpler way [2b, 3]. We know

already (see (9)) that the growing part of the function $\bar{\mu}/a$, in its lowest nonvanishing long-wavelength approximation, is a constant C_1 - this fact immediately follows from equation (2). If one likes, one is free to work with the function ζ , facing the same problem that we discuss here. One should give physical interpretation to the function ζ and realize that the “standard” formula predicts a huge value for this quantity at the second crossing, because $\zeta_f \approx (\delta\rho/\rho)_f$. Since $\zeta_f \approx \zeta_i$, this huge value should have been postulated from the very beginning, from the first crossing. Obviously, the ζ conservation law, which the authors [1] defend so strongly, does not allow any “big amplification” of ζ on the way from one crossing to another.

This appears to be the end of the story about constancy of ζ . However, we need to return to its beginning in view of the discussion in [1]. The fundamental equation often used in the literature is (see eq. (2.11) in [1]):

$$\Phi'' + 2(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'})\Phi' + [n^2 + 2(\mathcal{H}' - \mathcal{H}\frac{\varphi_0''}{\varphi_0'})]\Phi = 0. \quad (15)$$

The original derivation and use of the “constancy of ζ for superhorizon modes” was based on eq. (15) in which the term $n^2\Phi$ was neglected. Then, in terms of ζ defined by (12), this truncated equation is equivalent to $\zeta' = 0$ from which the “conservation law”

$$\zeta = \text{const} = X$$

was first derived (see for example [13]). It was shown [2b, 3] that the fundamental equation (2) requires this constant X to be a strict zero, this constant X is not determined by initial conditions for the function $\mu(\eta)$. The constant X arises because eq. (15) is a third-order differential equation in terms of $\mu(\eta)$. In the decomposition over small parameter n^2 this constant X would stand in front of n^{-2} : $\zeta(\eta) = X/2n^2 + \dots$. It is clearly stated [2b, 3] that it is this constant X that must be zero, not the entire function $\zeta(\eta)$, and not the constant C_1 which has a different origin and requires reference to the growing solution. Martin and Schwarz [1] essentially repeat all this analysis and come to the same conclusion, but, regrettably, portrait it as a Grishchuk’s “mistake”, rather than a complete agreement with his conclusion. After this clarification, we return to the “conserved” solutions determined by the constant C_1 .

The authors of [1] do not analyze the physical meaning of ζ and do not discuss its initial numerical value. They insist on the “nonemptiness” of the ζ conservation law (see also [14]) and “show how to make use of the constancy of ζ ”. They operate with the definition (12) and arrive at their central statement (4.25): “During inflation $w \sim -1$ and therefore the large amplification

$$\Phi_n(t_m) \simeq \frac{3}{5}\zeta \simeq \frac{2}{5} \frac{1}{1+w(t_i)} \Phi_n(t_i) \quad (4.25)$$

follows, which is the same as (4.7)”. Formula (4.7) is

$$\frac{\Phi_m}{\Phi_i} \simeq \frac{2}{5} \frac{2\beta_i + 3}{\beta_i + 1} \frac{1}{1+w_i} \approx \frac{2}{5} \frac{1}{1+w_i} \quad (4.7)$$

and is characterized as the “standard result” [1]. One more time Martin and Schwarz arrived at (4.7) in the form of their eq. (6.10). Apparently, the variant (6.10) is considered especially

valuable, because the authors of [1] note that they have now “obtained the ‘standard result’ (4.7) entirely within the synchronous gauge, without any reference to the constancy of ζ or the joining conditions of Deruelle and Mukhanov”. This last expression for the “large amplification” of Φ directly goes over into the (incorrect) final conclusion: “the closer the inflationary epoch is to the de Sitter space-time, the less important are large-scale gravitational waves in the CMBR today”. So, we need to sort out this “large amplification” of Φ .

Since we have in our hands a complete solution to the problem of evolution, it is not difficult to calculate the evolution of Φ as well. The function Φ is expressed in terms of the function $\bar{\mu}/a$ by (13). The dominant (“growing”) term of $\bar{\mu}/a$ is a constant C_1 (see (8)). So, in order to find the first nonvanishing approximation for Φ , we need to go to the next term in the long-wavelength decomposition of $\bar{\mu}/a$. It was shown [3] that the more accurate solution for $\bar{\mu}/a$ reads

$$\frac{\bar{\mu}}{a} = C_1 \left[1 - n^2 \int \frac{1}{a^2 \gamma} \left(\int a^2 \gamma d\eta \right) d\eta \right] + C_2 \int \frac{d\eta}{a^2 \gamma} + \dots \quad (16)$$

(this formula is given as (4.18) in [1]). Using (16) and neglecting the term with C_2 , we derive from (13) the expression for Φ which we will be working with:

$$\Phi = -\frac{1}{2} C_1 \frac{\alpha}{a^2} \int a^2 \gamma d\eta. \quad (17)$$

Let us first make a brief inspection of (17). It was shown [2] that the gauge-invariant potential Φ is strictly zero at the de Sitter stage, that is, $\Phi = 0$ for $\gamma = 0$. This fact is reflected in the exact formula (13) and in the approximate solution (17): the function $\Phi(\eta) = 0$ as long as $\gamma(\eta) = 0$. If $\gamma(\eta)$ is not strictly zero but is very small, $\gamma \ll 1$, $\Phi(\eta)$ also is expected to be very small. If $\gamma(\eta)$ is growing from zero, or almost zero, up to $\gamma = 2$ or $\gamma = 3/2$ (the cases of our interest) the function $\Phi(\eta)$ is also expected to grow. In other words, the absolute value of $\Phi(\eta)$ is expected to grow from zero or almost zero up to some finite number, which we will calculate shortly. The reheating transition can be either smooth or sharp, it does not matter. The integration of the possible step function $\gamma(\eta)$ presents no difficulty, the function (17) remains continuous. This is simply a reflection of the exact result mentioned before: the function v , eq. (11), and therefore (because α is continuous) the function Φ , eq. (13), must be continuous across a sharp transition. As expected, the first time derivative of Φ must experience a finite jump at the sharp transition. This follows from the exact relationship $\Phi' = -\alpha(\gamma+1)\Phi - (1/2)\alpha\gamma(\bar{\mu}/a)$ which is a consequence of (13) and (2). Since the functions $\bar{\mu}/a$ and Φ are continuous, the finite jump of Φ' is determined by the finite jump of γ . This finite jump of Φ' guarantees that even if the initial period of expansion was a strict de Sitter, the function $\Phi(\eta)$ would start slowly growing after the completion of that period, without developing any infinities, divergencies, or violations of linear perturbation theory, associated with the fact that $\gamma = 0$ at the initial stage. Correspondingly, the amplitudes in the Harrison-Zeldovich spectrum do not blow up to infinity. It was already emphasized [2] that there is nothing spectacular about the de Sitter case, the perturbation amplitudes remain finite and small.

Let us now calculate (17) at the interval of expansion described by a power-law scale factor

$$a(\eta) = l_o |\eta|^{1+\beta}. \quad (18)$$

In this case, $\gamma(\eta)$ is a constant: $\gamma = (2 + \beta)/(1 + \beta)$ and $\gamma = 0$ ($\beta = -2$) corresponds to the de Sitter solution. The function (17) reduces to

$$\Phi = -\frac{1}{2}C_1\gamma\frac{1+\beta}{3+2\beta}. \quad (19)$$

Strictly speaking, the integration constant in (17) produces an additional term proportional to α/a^2 , but this term is decaying and can be neglected, like the one (already neglected) which is associated with the constant C_2 in (16). The approximate solution (19) follows also directly from (13) and the exact solution for $\bar{\mu}/a$ which can be found in the power-law cases (18) [2].

To get a complete and concrete result, we will now evolve the function Φ through two power-law stages, and from one Hubble radius crossing to another. The first stage is

$$a_i = l_o |\eta|^{1+\beta_i}$$

the second stage is

$$a_m = l_m (\eta - \eta_e)^{1+\beta_m}. \quad (20)$$

The constants l_m and η_e are so chosen that $a(\eta)$ and $\alpha(\eta)$ are continuous at the transition point $\eta = \eta_1$. The cases of particular interest are $\gamma_i \ll 1$ and $\gamma_m = 2$ or $\gamma_m = 3/2$.

The function (17) takes the form

$$\Phi(\eta) = -\frac{1}{2}C_1\gamma_i\frac{1+\beta_i}{3+2\beta_i} - \frac{1}{2}C_1\gamma_m\frac{\alpha}{a^2}\int_{\eta_1}^{\eta} a^2 d\eta. \quad (21)$$

The first term in (21):

$$\Phi(\eta_i) = -\frac{1}{2}C_1\gamma_i\frac{1+\beta_i}{3+2\beta_i} \quad (22)$$

gives the value of $\Phi(\eta)$ at the interval of evolution from the first crossing $\eta = \eta_i$ and up to $\eta = \eta_1$. The second term in (21) gives the additional contribution at the interval of evolution from $\eta = \eta_1$ and up to the second crossing $\eta = \eta_m$. If the first stage is close to the interval of the de Sitter expansion, $\beta_i \approx -2$, $\gamma_i \ll 1$, then

$$\Phi(\eta_i) = -\frac{1}{2}C_1\gamma_i = -\frac{3}{4}C_1(1+w_i). \quad (23)$$

Taking the integral (21) with the help of (20) we arrive at the final value of $\Phi(\eta)$:

$$\Phi(\eta_m) = -\frac{1}{2}C_1\gamma_i\frac{1+\beta_i}{3+2\beta_i} - \frac{1}{2}C_1\gamma_m\frac{1+\beta_m}{3+2\beta_m} \left[1 - \left(\frac{\eta_1 - \eta_e}{\eta_m - \eta_e} \right)^{3+2\beta_m} \right].$$

Since $a_m(\eta_m) \gg a_m(\eta_1)$ and $\gamma_m \gg \gamma_i$, the $\Phi(\eta_m)$ simplifies:

$$\Phi(\eta_m) = -\frac{1}{2}C_1\gamma_m\frac{1+\beta_m}{3+2\beta_m}.$$

Specifically for the matter-dominated era, $\beta_m = 1$, $\gamma_m = 3/2$, we get

$$\Phi(\eta_m) = -\frac{3}{10}C_1. \quad (24)$$

Thus, the gauge-invariant potential $\Phi(\eta)$ grows from its value (23) up to its value (24). The final value of $\Phi(\eta)$ is of the order of C_1 , i. e. $\Phi(\eta)$ barely reaches the numerical level of the characteristic metric amplitude (10) but never exceeds it. The perturbation $\delta\rho/\rho_0$ in the matter density will also reach only the level C_1 .

The comparison of this calculation with the concept of the “large amplification” of scalar metric perturbations demonstrates that this concept is simply a misinterpretation. Imagine that we work with the function $f(x) = 10^{-20}\sin x$. Construct the ratio $f(x_m)/f(x_i)$ where $x_m = \pi/2$ and $x_i \ll 1$. The ratio is $1/x_i$, and it goes to infinity for $x_i \rightarrow 0$. It would be greatly misleading to think that the function $f(x)$ had experienced a very “large amplification”. The real meaning of this amplification is that the function $f(x)$ can go through zero, but can never exceed the level 10^{-20} .

The situation with the “large amplification” of $\Phi(\eta)$ due to reheating [1] is similar. The exact formula (14) for ζ generates $\zeta = -(1/2)C_1$ in the lowest nonvanishing approximation. The use of (23) and (24) shows that eq. (4.25) is trivially satisfied. The use of (24), (22) and (23) shows that eq. (4.7) is also trivially satisfied. [To check that the intermediate part of (6.10) is trivially satisfied, one needs to use the continuity of the function $\bar{\mu}/a$.] The particular way of evolution (as demonstrated above) of the gauge-invariant potential Φ is specific for this particular quantity. If, instead of Φ , we chose to work with a different potential, say $\bar{\Phi}$ where $\bar{\Phi} = \Phi/\gamma$, the idea of “large amplification” would not have even arisen. The quantity $\bar{\Phi}$ stays at the level C_1 at the first crossing and does not surpass this level during all the time up to the second crossing. The potential $\bar{\Phi}$ is more adequate for the problem, because $\bar{\Phi} = (1/2n^2)\alpha(\bar{\mu}/a)'$, see eq. (13), and $\bar{\Phi}$ plays the role of the generalized “momentum” conjugate to the generalized “coordinate” $\bar{\mu}/a$.

Thus, the real meaning of the “large amplification” (4.7), (4.25), (6.10) is not the one that the authors [1] want to assign to it, with the associated incredible (incorrect) prediction of the larger and larger today’s amplitude of density perturbations for smaller and smaller $1 + w_i$. The real meaning of this “large amplification” is the trivial fact that the function $\Phi(\eta)$ never goes above the level C_1 (the constant level of the scalar metric perturbation $h(\eta)$, see eq. (10)) even if $\Phi(\eta)$ has started from exceptionally small values at the almost de Sitter phase $w_i \approx -1$. [It seems that the authors [1] agree with this fact at least in one part of the paper, in the discussion at the end of Sec. IVA, but their final conclusion is in conflict with this discussion.] One of inflationary misuses is to write the ratio of today’s amplitudes for gravitational waves and density perturbations, in such a manner as if it was the contribution of gravitational waves which turned out to be, for some reason, exceptionally small in the limit $w_i \approx -1$ (see, for example, (7.1) in [1]). But the constancy of amplitude for gravitational waves has never been a matter of dispute. The reason for this small ratio is the claim about “big amplification” of the metric amplitude for density perturbations, on the route from the first crossing to the second crossing, which we have demonstrated to be incorrect.

After having sorted out all the questions of the evolution we need to return to the initial conditions which prescribe a numerical value to the constant C_1 . (The importance of initial conditions was emphasized in a recent work of Unruh [16].) We need to say a few words about quantum normalization.

The interest to primordial cosmological perturbations is related to the possibility of their generation from the inevitable zero-point quantum fluctuations by the mechanism of superadiabatic (parametric) amplification. This was first explored for the case of cosmological gravitational waves [10]. The very first concrete example, $a(t) \propto t$, $a(\eta) \propto e^\eta$, studied in [10] belongs to the class of background solutions which were later named inflationary. There is of course nothing specifically inflationary in the mechanism itself. The generated amplitudes and spectral slopes depend on the evolution of the very early Universe and can be used for inferences about its behaviour. Specifically for the power-law scale factors (18), and in the long-wavelength approximation, the spectral characteristic amplitude $h(n)$ (in logarithmic frequency intervals) was found to be $h(n) \approx (l_{Pl}/l_o)n^{2+\beta}$. The generated field exists today in the so-called squeezed vacuum quantum state and carries specific statistical properties, but these properties are outside of the scope of our present discussion. The quantum considerations are only relevant to our discussion as much as they determine the constant C_1 for gravity-wave solutions: $C_1 \approx (l_{Pl}/\lambda_i)$ (and $\lambda_i \approx l_o$ for models with the parameter β not far away from the de Sitter value $\beta = -2$).

Quantization of cosmological density perturbations is more complicated than quantization of gravitational waves, because the matter field fluctuations participate at the equal footing with the metric (gravitational field) fluctuations. Quantization of matter perturbations alone (for example, quantization of the scalar field perturbations in models considered here) would be as inconsistent as, say, quantization of the magnetic part of the electromagnetic field without quantization of its electric part. The model of a free quantized scalar field is useful to the extent that it allows us to understand the way in which the fundamental constants G , c , \hbar should participate in the final expression [2], but this model is certainly not sufficient for evaluation of the constant C_1 for scalar metric perturbations. After all, it is the metric (curvature) perturbations that we are mostly interested in. We have to deal with a combined degree of freedom, which we call the “fundamental” field [2]. A spatial Fourier component of this field is essentially the dimensionless function $\bar{\mu}/a$, and its value $(\bar{\mu}/a)_i$ should be evaluated from quantum considerations. For models with $\gamma = \text{const}$, that were actually considered [2], the constant $1/\sqrt{\gamma}$ combines with the constant in front of the relevant Bessel solution for $\mu(\eta)$, as well as with the value a_i of the scale factor $a(\eta)$ at the first crossing for a given mode, to produce the dimensionless constant of our interest C_1 for density perturbations. The order of magnitude normalization, based on the assigning of a half of the quantum to each mode of the “fundamental” field, gives $C_1 \approx (l_{Pl}/\lambda_i)$ in full analogy with gravitational waves. And the spectral characteristic amplitude $h(n)$ for the (scalar) metric perturbations is again $h(n) \approx (l_{Pl}/l_o)n^{2+\beta}$ [2, 3]. [More accurate calculation with exact solutions gives some numerical preference to gravitational waves [2, 3].]

If one does not want to deal with the “fundamental” field and prefers to quantize the (residual-gauge-invariant) scalar field perturbations which include metric perturbations, one is free to do that. The function that will do the job is $\psi_1 = \varphi_1/\sqrt{1+w} + (3/4\kappa)^{1/2}h$. Because of the exact relationship [2] $\varphi_1(\eta) = (2\kappa)^{-1/2}[\mu/a - \sqrt{\gamma}h]$, this function is $\psi_1 = (3/4\kappa)^{1/2}(\bar{\mu}/a)$. On the other hand, the function ψ_1 is the gauge-invariant scalar field variable

(called for example in the recent paper [17] by $\varphi_{\delta\phi}$) calculated in the synchronous “gauge”. It is seen from these expressions that it would be incorrect to naively prescribe the quantum normalization to φ_1 on the grounds of the free scalar field quantization: $\varphi_1 \sim H_i$. If one takes this evaluation of the “vacuum fluctuations” and then sends $1 + w$ to zero, the value of $(\psi_1)_i$, as well as the metric perturbation h_i , become arbitrarily large already at the first crossing in proportion to $1/\sqrt{1 + w_i}$.

It is necessary to mention that sometimes one tries to make sense of the “standard” formula for $(\delta\rho/\rho)_f$ by saying that although there is not any “big amplification” on the way from one crossing to another, the formula itself applies only to cases $(\dot{\varphi})_i \gg H_i$, that is, to $\sqrt{1 + w_i} \gg 1$ and $\sqrt{\gamma_i} \gg 1$. If so, all the claims about dominance of density perturbations over gravitational waves are false from the very beginning, because they all are based on the opposite inequalities. In addition, this position seems to be a refusal to answer the question on what will happen in models with the opposite inequalities. The present author believes that there is no problem with quantization in those models as well, and that they were correctly treated in [2, 3]. In any case, the idea of a “big amplification” [1] in course of evolution, as an explanation for the “standard” formula, is incorrect and does not help in resolution of a possible new dispute - this time, over quantization.

The conclusions of [2, 3] are relevant not only for the interpretation of the observed CMB anisotropies, but also, and may be more importantly, for the prospects of detecting relic gravitational waves by terrestrial and cosmic instruments [15].

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